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Abstract: We investigate the asymptotic distribution of the Exponential Tail (ET) estimator of extreme quantiles. We give sufficient conditions for the asymptotic normality and provide some illustrating examples. Then, on the basis of this result, we propose a goodness-of-fit test for the tail of a usual distribution. The asymptotic power and level of the test are established.

Key-words: Exponential Tail, Extreme quantiles, Asymptotic distribution, Second order conditions, Goodness-of-fit test.

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Normalité asymptotique de la méthode ET pour l'estimation des quantiles extrêmes. Application au test ET

Résumé : Nous étudions la loi asymptotique de la méthode ET (Exponential Tail) pour l'estimation des quantiles extrêmes. Nous donnons des conditions suffisantes pour obtenir la normalité asymptotique de l'estimateur. Ce résultat est illustré sur des classes de lois classiques. Nous mettons alors à profit la normalité asymptotique de l'estimateur ET pour construire un test d'adéquation à la queue d'une loi usuelle. Le niveau et la puissance asymptotiques du test sont établis.

Mots-clés : Queue exponentielle, Quantiles extrêmes, Loi asymptotique, Conditions du second ordre, Test d'adéquation.

1 Introduction.

We establish the asymptotic distribution of the ET (Exponential Tail) estimator, a non-parametric estimator of the extreme quantiles of an unknown distribution. We take profit of this result to propose a goodness-of-fit test for the tail of a usual distribution. Given an n-sample $X_1, \dots, X_n$ from a cumulative distribution function F with associated survival distribution function $\bar{F} = 1 - F$, an extreme quantile is a $(1-p_n)$-th quantile x_{p_n} of F essentially larger than the maximal observation, i.e. such that $\bar{F}(x_{p_n}) = p_n$ with $p_n \leq 1/n$. The estimation of extreme quantiles requires specific methods. The POT (Peaks Over Threshold) method relies on an approximation of the distribution of excesses over a given threshold [18]. More precisely, let us introduce deterministic thresholds u_n such that $\bar{F}(u_n) = m_n/n$ with

$$1 \leq m_n \leq n, \quad m_n \to \infty, \quad m_n/n \to 0 \text{ as } n \to \infty. \tag{1}$$

The excesses above u_n are defined on the basis of the $X_i > u_n$'s by $Y_i = X_i - u_n$. The random number of exceedances is thus given by $N_n = \sum_{i=1}^{n} \mathbb{I}\{X_i > u_n\}$, and the survival distribution function of the excesses is defined by $\bar{F}_{u_n}(x) = \bar{F}(u_n + x)/\bar{F}(u_n)$. Pickands' theorem [13, 11] states that under some conditions, $\bar{F}_{u_n}$ can be approximated by a Generalized Pareto Distribution (GPD). As a consequence, the extreme quantile x_{p_n} can be approximated by the deterministic term

$$x_{\mathrm{GPD},\,n} = u_n + \frac{\sigma(u_n)}{\gamma(u_n)} \left[r_n^{\gamma(u_n)} - 1 \right], \tag{2}$$

with $r_n = m_n/(np_n)$ and where $\sigma(u_n)$ and $\gamma(u_n)$ are respectively the scale and shape parameters of the GPD distribution. Then, the POT method consists of estimating these two unknown parameters. The ET method corresponds to the important particular case where F is known to belong to Gumbel's Maximum Domain of Attraction, DA(Gumbel). In such a situation, $\gamma(u_n) = 0$ and the GPD distribution reduces to an exponential distribution with scale parameter $\sigma(u_n)$. Thus, approximation (2) can then be rewritten as

$$x_{\mathrm{ET},\,n} = u_n + \sigma(u_n) \ln(r_n)$$

and the corresponding estimator [3] is

$$\widehat{x}_{\mathrm{ET},\,n} = u_n + \widehat{\sigma}_n \ln\left(\frac{N_n}{np_n}\right), \tag{3}$$

where $\widehat{\sigma}_n$ is the empirical mean of the N_n excesses: $\widehat{\sigma}_n = N_n^{-1} \sum_{i=1}^{N_n} Y_i$. The reasons motivating our study of the asymptotic properties of the ET estimator are both theoretical and practical. From a practical point of view, the ET estimator realizes a good balance between simplicity and generality, since most of the standard distributions (normal, Weibull, lognormal, gamma, exponential) belong to DA(Gumbel). Moreover, the ET estimate does not require estimation of the shape parameter $\gamma(u_n)$ which is difficult in practice (see, e.g., [10], pages 327–356 for a review). For this reason, the asymptotic properties of the

ET estimator cannot be directly derived from those of the POT estimator (see Section 2, Remark 2.2). A specific study is necessary. Our result is the following. Under suitable conditions,

$$\frac{m_n^{1/2}}{\sigma(u_n)\ln(r_n)}\left(\widehat{x}_{\mathrm{ET},\,n}-x_{\mathrm{ET},\,n}\right)\xrightarrow{d}N(0,1)\text{ as }n\to\infty. \tag{4}$$

It is possible to take profit of this result to build a goodness-of-fit test to the distribution tail. More precisely, we wish to check that a model $F_0 \in \mathrm{DA}(\mathrm{Gumbel})$ provides an acceptable approximation to the distribution above the maximal observation. To this end, an approximate $\alpha-$confidence interval $CI_{\alpha,n}$ for the extreme quantile x_{p_n} is deduced from (4). The assumption $\{F = F_0\}$ is rejected if the parametric estimator $x_{\mathrm{param},\,n} = \bar{F}_0^{-1}(p_n)$ does not belong to the confidence interval $CI_{\alpha,n}$.

This paper is organized as follows: in Section 2, we precise the result (4) and provide some examples and comments. The goodness-of-fit test is described in Section 3 and its main properties are established. Auxiliary results are given in Section 4 and the proofs are postponed to the Appendix.

2 Main result.

We will need the following assumptions throughout the paper:

(A1): F is twice differentiable and has an inverse function F^{-1}.

Under **(A1)**, we define $V(x) = \bar{F}^{-1}(e^{-x})$, $A(x) = V''/V'(\ln x)$, $a(x) = V''/V'(V^{-1}(x))$ and we assume the following condition:

(A2): $A(x) \to 0$ as $x \to \infty$, A has asymptotically a constant sign and $\exists \rho \leq 0$, $|A| \in \mathcal{RV}_\rho$.

We recall that $\mathcal{RV}_\rho$ is the set of regularly varying functions with index ρ (see, e.g., [2] for more details). This kind of condition has been introduced in [5] as a "second order von Mises condition". Finally, we need two further conditions on the sequences (m_n) and (p_n):

(A3): $m_n/(np_n) \to \infty$ as $n \to \infty$,

(A4): $m_n^{1/2} a(u_n) \to 0$ as $n \to \infty$.

Our main result is the following (see the Appendix for a proof):

Theorem 1 *Suppose (1) is verified. Then, under* **(A1)**–**(A4)**,

$$\frac{m_n^{1/2}}{\sigma(u_n)\ln(r_n)}\left(\widehat{x}_{ET,\,n}-x_{ET,\,n}\right)\xrightarrow{d}N(0,1)\ \text{as } n\to\infty,$$

where $\sigma(u_n) = V'(V^{-1}(u_n))$.

Remark 2.1 *We consider the estimator $\widehat{x}_{ET,n}$ defined by a deterministic threshold u_n and a random number of exceedances N_n. The dual approach would consider a deterministic number of exceedances m_n and a random threshold $X_{(n-m_n,\,n)}$, where $X_{(1,n)} \le \cdots \le X_{(n,n)}$ are the order statistics associated to $X_1, \ldots, X_n$. The corresponding ET estimator can then be written*

$$\tilde{x}_{ET,\,n} = X_{(n-m_n,\,n)} + \widehat{\sigma}_n \ln(r_n)$$

and $\tilde{x}_{ET,\,n}$ has the same limiting distribution as $\widehat{x}_{ET,\,n}$:

$$\frac{m_n^{1/2}}{\sigma(u_n)\ln(r_n)} \left(\tilde{x}_{ET,\,n} - x_{ET,\,n}\right) \xrightarrow{d} N(0,1) \ \text{as } n \to \infty. \tag{5}$$

The proof is similar to the proof of Theorem 1. We condition on $X_{(n-m_n,\,n)}$ rather than N_n (see Appendix, proof of Lemma 4.2).

Remark 2.2 *In [6], Proposition 1, sufficient conditions are given for*

$$\frac{m_n^{1/2}}{x_{p_n}} \left(\tilde{x}_{ET,\,n} - x_{p_n}\right) \xrightarrow{d} N(0,\,1) \ \text{as } n \to \infty. \tag{6}$$

This convergence is derived from a similar result on the asymptotic behavior of the POT estimate. Results (5) and (6) are not equivalent, and (6) does not seem to work in all cases. Let us consider the example where F is the standard exponential distribution function. In such a case $\sigma(u_n) = 1$, $x_{ET,\,n} = x_{p_n}$ and then

$$\frac{m_n^{1/2}}{x_{p_n}} \left(\tilde{x}_{ET,\,n} - x_{p_n}\right) = \frac{1}{\ln(1/p_n)} \left[m_n^{1/2}(X_{(n-m_n,\,n)} - u_n) + \ln(r_n) m_n^{-1/2} \sum_{j=1}^{m_n} (Y_j - 1) \right]. \tag{7}$$

We can see that these random variables converge to 0 in probability: First, $m_n^{1/2}(X_{(n-m_n,\,n)} - u_n)$ is asymptotically $N(0,\,1)$, see for instance [1], Theorem 8.5.3. Besides, the excesses Y_j, $1 \le j \le m_n$, are i.i.d. $Exp(1)$, hence by the Central Limit Theorem $m_n^{-1/2} \sum_{j=1}^{m_n} (Y_j - 1)$ is asymptotically $N(0,\,1)$ as well. This shows that (7) converges to 0 in probability since $\ln(r_n)/\ln(1/p_n) \to 0$ as $n \to \infty$ under the conditions of [6], Proposition 1. On the contrary, (5) gives the correct result. Simulations confirm this difference. 100 samples of size $n = 10{,}000$ were simulated. We chose $p_n = 1/(n \ln n)$ and $m_n = \ln n$, and computed both normalizations (5) and (6). Figure 1 represents the corresponding empirical distributions. It appears that (6) is not a valid normalization, since it leads to convergence to a Dirac distribution rather than convergence to a non degenerate normal distribution. On the contrary, the normalization (5) seems to work, even though convergence toward the normal distribution is very slow.

In order to obtain an explicit form for our sufficient conditions, it is convenient to introduce three subclasses of DA(Gumbel) (see [8, 14]): $V \in \mathcal{C} = \mathcal{C}_\theta^1 \cup \mathcal{C}^2 \cup \mathcal{C}_\theta^3$ with

$\mathcal{C}^1_\theta = \mathcal{SR}_{1/\theta}$, $\theta > 0$, $\theta \neq 1$,

$\mathcal{C}^1_1 = \mathcal{C}^1_{1,\infty} \cup \mathcal{C}^1_{1,\tau} = \{V \in \mathcal{SR}_1 \ : \ V'' = 0\} \cup \{V \in \mathcal{SR}_1 \ : \ |V''| \in \mathcal{SR}_{-1-\tau}\}$, $\tau \geq 0$,

$\mathcal{C}^2 = \{V \in \mathcal{SR}_0,\ V' \in \mathcal{SR}_{-1}\}$,

$\mathcal{C}^3_\theta = \{V = \exp g,\ g \in \mathcal{SR}_\theta,\ 0 < \theta < 1\}$,

where $\mathcal{SR}_\theta$ is the subset of smooth regularly varying functions in $\mathcal{RV}_\theta$, [2]. For instance, $\mathcal{C}^1_\theta$ ($\theta \neq 1$) includes normal ($\theta = 2$) and Weibull distributions ($\theta = \beta$, the shape parameter). The subclass $\mathcal{C}^1_{1,\infty}$ includes exponential distributions and the subclass $\mathcal{C}^1_{1,\tau}$ includes gamma distributions ($\tau = 1$). The subclass $\mathcal{C}^2$ includes the double exponential distribution [1]. The subclass $\mathcal{C}^3_\theta$ contains the lognormal distribution ($\theta = 1/2$).

Corollary 2.1 *Suppose (1) is verified. If $m_n/(np_n) \to \infty$ as $n \to \infty$ and the following conditions hold:*

(i) $V \in \mathcal{C}^1_\theta \cup \mathcal{C}^2$, $\theta \neq 1$ and $m_n = o((\ln n)^2)$,
(ii) $V \in \mathcal{C}^1_{1,\infty}$,
(iii) $V \in \mathcal{C}^1_{1,\tau}$ and $m_n = O((\ln n)^{2(1+\tau)-\delta})$ $\forall \delta > 0$ arbitrary small,
(iv) $V \in \mathcal{C}^3_\theta$ and $m_n = O((\ln n)^{2(1-\theta)-\delta})$ $\forall \delta > 0$ arbitrary small.

then, when $n \to \infty$,

$$\frac{m_n^{1/2}}{\sigma(u_n)\ln(r_n)}\left(\widehat{x}_{ET,\,n} - x_{ET,\,n}\right) \xrightarrow{d} N(0,1).$$

The proof is straightforward since Lemma 4.4 allows to verify **(A4)**.

Remark 2.3 *Theorem 1 captures the behavior of the random variable $\widehat{x}_{ET,\,n} - x_{ET,\,n}$. The bias term $\delta_n = x_{ET,\,n} - x_{p_n}$ can be studied in the particular case where $V \in \mathcal{C}$. For instance, in [8], Lemma 3, it is shown that $\delta_n = d_n(1 + o(1))$ where*

$$d_n = \frac{1}{2}(\ln r_n)^2 V''(\varrho_n),$$

with $\varrho_n \in [\ln(1/p_n), \ln(n/m_n)]$. See also [16].

3 Application: The ET test.

This test is motivated by questions arising in the field of Structural Reliability, where the usual distributions are in DA(Gumbel), and have infinite endpoint. Given the sample $X_1, \ldots, X_n$, we wish to check that a model $F_0 \in$ DA(Gumbel) provides an acceptable approximation to the tail of the distribution in the range near and above the maximal observation. This is important since in the context of Structural Reliability events with low

[1] X has a double exponential distribution if $\exp X$ has an exponential distribution

probabilities can imply strong consequences as critical failures or extreme charges. We assume that at reasonable significance levels, usual goodness-of-fit tests have not rejected the null hypothesis $\mathcal{H}_0$ that $\{F = F_0\}$. Such procedures essentially test the adequacy of the model to the central range of the sample, that is to say the central part of the sample interval. The dangers of extrapolating in the tails from the results of such tests are detailed, e.g., in [9] and in [12]. The purpose of the ET test is to check the adequacy of the tail of a given F_0 to extreme observations and to check that this tail provides reasonable extrapolations above the maximal observation. Therefore, we wish to test

$$\mathcal{H}_0 : \{F = F_0\} \quad \text{against} \quad \mathcal{H}_1 : \{F = F_1\}$$

in the upper tail. The principle of the ET test is to compare two different estimates of some extreme quantile. The first one is the estimate under $\mathcal{H}_0$,

$$x_{\mathrm{param},\,n} = \bar{F}_0^{-1}(p_n)\,.$$

The second one is the ET estimate $\widehat{x}_{\mathrm{ET},\,n}$ defined by (3). When we estimate the true quantile by its ET estimation, we make two kinds of errors: an estimation error and an approximation error, since we approximate the excess distribution with an exponential distribution. Under $\mathcal{H}_0$, adding an asymptotic equivalent $d_{0,n}$ of the approximation error $\delta_{0,n}$ (see Remark 2.3) to the bounds of the confidence interval based on the asymptotic distribution of $\widehat{x}_{\mathrm{ET},\,n}$ (see Theorem 1) yields an approximate confidence interval for the true quantile:

$$CI_{\alpha,\,n} = \left[\widehat{x}_{\mathrm{ET},\,n} + d_{0,n} \pm \hat{\sigma}_n \ln(r_n) m_n^{-1/2} z_\alpha\right],$$

where z_α is such that $P(|\xi| > z_\alpha) = \alpha$ with $\xi \sim N(0,1)$. The ET goodness-of-fit test rejects $\mathcal{H}_0$ when the parametric estimate does not lay within this confidence interval: $x_{\mathrm{param},\,n} \notin CI_{\alpha,n}$. This is a simplified version of the ET test proposed in [7]. The full version aims to test

$$\mathcal{H}_0 : \{F = F_\theta\} \quad \text{against} \quad \mathcal{H}_1 : \{F \neq F_\theta\}$$

where θ is estimated by the maximum likelihood method. Besides, it is also proposed to replace the asymptotic Gaussian approximation of $\widehat{x}_{\mathrm{ET},\,n} - x_{\mathrm{param},\,n}$ by bootstrap.
For the sake of simplicity (see [8]), we assume that

$$\begin{aligned} &m_n \;\text{ is the largest integer } \leq \mathrm{cst}_1\, n^{1-p}(\ln n)^{-q} \quad \text{and} \\ &p_n \;=\; \mathrm{cst}_2\, n^{-p'}(\ln n)^{-q'} \end{aligned} \tag{8}$$

for some positive constants cst_1 and cst_2, where $0 < p \leq 1$ and $q > 0$ when $p = 1$ (with no constraint on q when $0 < p < 1$), and $p' \geq 1$ and $q' \geq 0$ when $p' = 1$ (with no constraint on q' when $p' > 1$).
Introducing $V_0(x) = \bar{F_0}^{-1}(e^{-x})$ and $V_1(x) = \bar{F_1}^{-1}(e^{-x})$, we have the following two results:

Theorem 2 *Consider two sequences (m_n) and (p_n) verifying (8). In each of the following cases:*

(i) $V_0 \in \mathcal{C}^1_\theta \cup \mathcal{C}^2$, $\theta \neq 1$, $p = p' = 1$ *and* $q < \min(2, q')$,

(ii) $V_0 \in \mathcal{C}^1_{1,\infty}$, $p < p'$ *or* $q < q'$,

(iii) $V_0 \in \mathcal{C}^1_{1,\tau}$, $p = p' = 1$ *and* $q < \min(2(1+\tau), q')$,

(iv) $V_0 \in \mathcal{C}^3_\theta$, $p = p' = 1$ *and* $q < \min(2(1-\theta), q')$,

the level of the ET test converges to α *as* $n \to \infty$.

Theorem 3 *Consider two sequences (m_n) and (p_n) verifying (8). In each of the following cases:*

(i) $V_0 \in \mathcal{C}^1_\theta$, $V_1 \in \mathcal{C} \setminus \mathcal{C}^1_\theta$, $\theta \neq 1$, $p = p' = 1$ *and* $q < \min(2, q')$,

(ii) $V_0 \in \mathcal{C}^1_{1,\tau}$, $V_1 \in \mathcal{C} \setminus \mathcal{C}^1_{1,\tau}$, $p = p' = 1$ *and* $q < \min(2(1+\tau), q')$,

(iii) $V_0 \in \mathcal{C}^1_{1,\infty}$, $V_1 \in \mathcal{C} \setminus \mathcal{C}^1_{1,\infty}$, $p < p'$ *or* $q < q'$,

(iv) $V_0 \in \mathcal{C}^2$, $V_1 \in \mathcal{C} \setminus \mathcal{C}^2$, $p = p' = 1$ *and* $q < \min(2, q')$,

(v) $V_0 \in \mathcal{C}^3_\theta$, $V_1 \in \mathcal{C} \setminus \mathcal{C}^3_\theta$, $p = p' = 1$ *and* $q < \min(2(1-\theta), q')$,

the power of the ET test converges to 1 as $n \to \infty$.

4 Auxiliary results.

To prove Theorem 1, we need three lemmas. They control the asymptotic behavior of the random variables N_n and $\widehat{\sigma}_n$ from which $\widehat{x}_{\mathrm{ET},\,n}$ is defined.

Lemma 4.1 *Suppose (1) is verified. Then,* $m_n^{-1/2}(N_n - m_n) \xrightarrow{d} N(0,\,1)$ *as* $n \to \infty$.

From [17], Lemma 1.8, $\bar{F}$ is integrable, hence so is $\bar{F}_{u_n}$ and we introduce

$$\mu(u_n) = \int_0^\infty \bar{F}^\sharp_{u_n}(y)\,dy, \tag{9}$$

where $\bar{F}^\sharp_{u_n}(y) = \bar{F}_{u_n}(\sigma(u_n)y)$ is the survival function of normalized excesses.

Lemma 4.2 *Under* **(A1)**, $N_n^{1/2}(\widehat{\sigma}_n/\sigma(u_n) - \mu(u_n)) \xrightarrow{d} N(0,1)$ *as* $n \to \infty$.

Lemma 4.3 *Under* **(A1)**, **(A2)** *and* **(A4)**, $m_n^{1/2}(\mu(u_n) - 1) \to 0$ *as* $n \to \infty$.

The next lemma is useful to rewrite condition **(A4)** in a more explicit form for the considered classes of distributions. It states that V''/V', which plays a central role in this study, is a smooth regularly varying function provided $V \in \mathcal{C}$. Its proof is straightforward.

Lemma 4.4 *(i) If* $V \in \mathcal{C}^1_\theta \cup \mathcal{C}^2$ $(\theta \neq 1)$ *then* $V''/V' \in \mathcal{SR}_{-1}$.
(ii) If $V \in \mathcal{C}^1_{1,\infty}$ *then* $V''/V' = 0$.
(iii) If $V \in \mathcal{C}^1_{1,\tau}$ *then* $V''/V' \in \mathcal{SR}_{-1-\tau}$.
(iv) If $V \in \mathcal{C}^3_\theta$ *then* $V''/V' \in \mathcal{SR}_{\theta-1}$.

The next lemma will reveal useful to apply Lemma 4.4.

Lemma 4.5 *Suppose* $V \in \mathcal{C}$, *and* $\ln x/\ln\zeta \to 0$ *as* $\zeta \to 0$. *Then, for* $\zeta \to 0$,

$$\frac{V''(\kappa_2 - \ln\zeta - \kappa_3(\kappa_2 + \kappa_1 \ln x))}{V'(\kappa_2 - \ln\zeta)} \sim \frac{V''(-\ln\zeta)}{V'(-\ln\zeta)}, \quad \forall\, \kappa_1,\, \kappa_2,\, \kappa_3\, \in [0,1]. \tag{10}$$

The study of the asymptotic level and power of the ET test requires further notations. For $k \in \{0,1\}$ we introduce:

- The true quantile under $\mathcal{H}_k$: $x_{k,p_n} = \bar{F}_k^{-1}(p_n) = V_k(b_n)$ with $b_n = \ln(1/p_n)$,
- Its ET approximation under $\mathcal{H}_k$: $x_{\mathrm{ET},\,k,n} = u_{n,k} + \sigma_k(u_{k,n})\ln(r_n)$ where $u_{n,k} = \bar{F}_k^{-1}(n/m_n) = V_k(a_n)$ with $a_n = \ln(n/m_n)$, and $\sigma_k(u_{k,n}) = V_k'(V_k^{-1}(u_{k,n}))$,
- The corresponding error of approximation under $\mathcal{H}_k$: $\delta_{k,n} = x_{k,p_n} - x_{\mathrm{ET},\,k,n}$,
- Its first order approximation under $\mathcal{H}_k$ (see Remark 2.3): $d_{k,n} = (\ln r_n)^2 V_k''(\varrho_n)/2$ with
$\varrho_n \in [b_n, a_n]$,
- The auxiliary function $\Omega_k(t) = tV_k'(t)/V_k(t)$, $t > 0$.

We also define $\Omega(t) = V_0(t)/V_1(t)$, $t > 0$, and the following sequences

$$\omega_n = \sqrt{m_n}\frac{d_{0,n} - \delta_{0,n}}{\sigma_0(u_{0,n})\ln(r_n)}, \quad \phi_n = \sqrt{m_n}\frac{x_{\mathrm{ET},\,1,n} - x_{\mathrm{ET},\,0,n}}{\sigma_1(u_{1,n})\ln(r_n)} \quad \text{and} \quad \psi_n = \frac{\sigma_0(u_{0,n})}{\sigma_1(u_{1,n})}. \tag{11}$$

The next two lemmas describe the asymptotic behavior of these sequences which drive the asymptotic level and power of the ET test.

Lemma 4.6 *Under the conditions of Theorem 2,* $\omega_n \to 0$ *as* $n \to \infty$.

Lemma 4.7 *Consider two sequences* (m_n) *and* (p_n) *verifying (8). In each of the following cases:*

(i) $V_0 \in \mathcal{C}^1_\theta$, $V_1 \in \mathcal{C} \setminus \mathcal{C}^1_\theta$,

(ii) $V_0 \in \mathcal{C}^2$, $V_1 \in \mathcal{C} \setminus \mathcal{C}^2$,

(iii) $V_0 \in \mathcal{C}^3_\theta$, $V_1 \in \mathcal{C} \setminus \mathcal{C}^3_\theta$,

we have $|\phi_n| \to \infty$ *and* $\psi_n = o(\phi_n)$ *as* $n \to \infty$.

Appendix: Proof of theorems.

Proof of Theorem 1

Let us denote

$$\Delta_n = \frac{m_n^{1/2}}{\sigma(u_n)\ln r_n}\left(\widehat{x}_{\mathrm{ET},\,n} - x_{\mathrm{ET},\,n}\right)$$

and split it into four terms, $\Delta_n = \Delta_n^{(a)} + \Delta_n^{(b)} + \Delta_n^{(c)} + \Delta_n^{(d)}$. Each term is treated separately:

- $\Delta_n^{(a)}$ is the stochastic term defined by $\Delta_n^{(a)} = N_n^{1/2}(\widehat{\sigma}_n/\sigma(u_n) - \mu(u_n))$ and studied in Lemma 4.2. It converges in distribution to the centered and reduced normal distribution under **(A1)**.

- $\Delta_n^{(b)}$ is a stochastic term defined by $\Delta_n^{(b)} = [(N_n/m_n)^{-1/2} - 1]\Delta_n^{(a)}$. As a consequence of Lemma 4.1, N_n/m_n converges to 1 in probability. Therefore, $\Delta_n^{(b)} = o_P(\Delta_n^{(a)})$ and converges to 0 in probability under **(A1)**.

- $\Delta_n^{(c)}$ is a stochastic term defined by

  $$\Delta_n^{(c)} = m_n^{1/2}\frac{\widehat{\sigma}_n}{\sigma(u_n)}\frac{\ln(N_n/m_n)}{\ln(r_n)}.$$

  From Lemma 4.2, we can write $\widehat{\sigma}_n/\sigma(u_n) = \mu(u_n) + N_n^{-1/2}\xi_n$ with $\xi_n \xrightarrow{d} \xi \sim N(0,1)$. Besides, from Lemma 4.1, $N_n^{-1/2} \to 0$ in probability and from Lemma 4.3, $\mu(u_n) \to 1$ as $n \to \infty$. Therefore, $\widehat{\sigma}_n/\sigma(u_n)$ is bounded in probability and consequently

  $$\Delta_n^{(c)} = O_P\left(m_n^{1/2}\frac{\ln(N_n/m_n)}{\ln(r_n)}\right).$$

  In view of Lemma 4.1, we write $N_n/m_n = 1 + m_n^{-1/2}\pi_n$, with $\pi_n \xrightarrow{d} \pi \sim N(0,\,1)$. It follows that $\ln(N_n/m_n) = m_n^{-1/2}\pi_n + O_P(m_n^{-1})$ and

  $$\Delta_n^{(c)} = O_P\left(\frac{\pi_n}{\ln(r_n)}\right) = O_P\left(\frac{1}{\ln(r_n)}\right),$$

  which converges to 0 in probability under **(A3)**.

- $\Delta_n^{(d)}$ is the deterministic term defined by $\Delta_n^{(d)} = m_n^{1/2}(\mu(u_n) - 1)$ and considered in Lemma 4.3. It converges to 0 under **(A1)**, **(A2)** and **(A4)**.

Proof of Theorem 2

From Theorem 1, and under $\mathcal{H}_0$,

$$x_{\text{param},\,n} - \widehat{x}_{\text{ET},\,n} = x_{\text{param},\,n} - x_{\text{ET},\,0,n} + \sigma_0(u_{0,n})\ln(r_n)m_n^{-1/2}\ \xi_n$$

where $\xi_n \xrightarrow{d} \xi \sim N(0,1)$. Therefore,

$$P(x_{\text{param},\,n} \notin CI_{\alpha,n} \mid \mathcal{H}_0) = P\left(\xi_n \notin J_{\alpha,n}\right),$$

where $J_{\alpha,n} = [\omega_n \pm z_\alpha \widehat{\sigma}_n/\sigma_0(u_{0,n})]$ and ω_n is defined by (11). Now, since $m_n \to \infty$ and $m_n/n \to 0$, under $\mathcal{H}_0$ we have $\widehat{\sigma}_n/\sigma_0(u_{0,n}) \xrightarrow{P} 1$ as $n \to \infty$ [4]. We thus have

$$\mathrm{P}\left(\widehat{\sigma}_n/\sigma_0(u_{0,n}) > 1+\eta\right) \to 0 \quad \text{and} \quad \mathrm{P}\left(\widehat{\sigma}_n/\sigma_0(u_{0,n}) < 1-\eta\right) \to 0,$$

when $n \to \infty$ for all $\eta > 0$. Introducing $K_{\alpha,\,n} = [\omega_n \pm z_\alpha(1+\eta)]$, it follows that $\forall \eta > 0$,

$$\mathrm{P}\left(J_{\alpha,n} \subset K_{\alpha,\,n}\right) = \mathrm{P}\left(\widehat{\sigma}_n/\sigma_0(u_{0,n}) \le 1+\eta\right) = 1 - \mathrm{P}\left(\widehat{\sigma}_n/\sigma_0(u_{0,n}) > 1+\eta\right) \to 1. \tag{12}$$

From Lemma 4.6, $\omega_n \to 0$ as $n \to \infty$, and therefore there exists $N_0(\eta) \in \mathbb{N}$ such that $\forall n \ge N_0(\eta)$, $K_{\alpha,n} \subset [\pm z_\alpha(1+2\eta)]$ leading to

$$\mathrm{P}\left(J_{\alpha,n} \subset K_{\alpha,\,n}\right) \le \mathrm{P}\left(J_{\alpha,n} \subset [\pm z_\alpha(1+2\eta)]\right). \tag{13}$$

As a consequence of (12) and (13), $\mathrm{P}(J_{\alpha,n} \subset [\pm z_\alpha(1+2\eta)]) \to 1$ as $n \to \infty$. This is equivalent to

$$\forall \nu_1 > 0,\ \exists N_1(\nu_1,\eta) \in \mathbb{N} \ \text{ such that } \forall n \ge N_1(\nu_1,\eta), \quad \mathrm{P}\left(J_{\alpha,n} \not\subset [\pm z_\alpha(1+2\eta)]\right) < \nu_1.$$

It follows that $\forall n \ge N_1(\nu_1,\eta)$,

$$\begin{aligned}
\mathrm{P}\left(\xi_n \in J_{\alpha,n}\right) &= \mathrm{P}\left(\{\xi_n \in J_{\alpha,n}\} \ \cap \ \{J_{\alpha,n} \subset [\pm z_\alpha(1+2\eta)]\}\right) \\
&+ \mathrm{P}\left(\{\xi_n \in J_{\alpha,n}\} \ \cap \ \{J_{\alpha,n} \not\subset [\pm z_\alpha(1+2\eta)]\}\right) \\
&\le \mathrm{P}\left(\xi_n \in [\pm z_\alpha(1+2\eta)]\right) + \mathrm{P}\left(J_{\alpha,n} \not\subset [\pm z_\alpha(1+2\eta)]\right) \\
&\le \mathrm{P}\left(\xi_n \in [\pm z_\alpha(1+2\eta)]\right) + \nu_1.
\end{aligned} \tag{14}$$

Moreover, for all $\nu_2 > 0$, there exists $C(\nu_2) > 0$ and $N_2(\nu_2)$ such that

$$\forall \eta \in]0, C(\nu_2)] \ \text{ and } \ \forall n \ge N_2(\nu_2), \quad |\mathrm{P}(\xi_n \in [\pm z_\alpha(1+2\eta)]) - (1-\alpha)| < \nu_2,$$

which implies that

$$\forall \nu_2 > 0, \quad \forall \eta \in]0, C(\nu_2)], \quad \limsup_{n\to\infty} \mathrm{P}(\xi_n \in [\pm z_\alpha(1+2\eta)]) \le 1-\alpha+\nu_2. \tag{15}$$

From (14) and (15), we obtain $\forall \nu_1,\ \nu_2 > 0$

$$\limsup_{n\to\infty} \mathrm{P}\left(\xi_n \in J_{\alpha,n}\right) \le 1-\alpha+\nu_1+\nu_2.$$

Similarly, it can be shown that $\forall \nu_1,\ \nu_2 > 0$,

$$\liminf_{n\to\infty} \mathrm{P}\left(\xi_n \in J_{\alpha,n}\right) \geq 1 - \alpha - \nu_1 - \nu_2.$$

This entails: $\forall \nu_1,\ \nu_2 > 0$,

$$1 - \alpha - \nu_1 - \nu_2 \leq \liminf_{n\to\infty} \mathrm{P}(\xi_n \in J_{\alpha,n}) \leq \limsup_{n\to\infty} \mathrm{P}(\xi_n \in J_{\alpha,n}) \leq 1 - \alpha + \nu_1 + \nu_2.$$

Since ν_1 are ν_2 are arbitrarily small, both limits are equal and thus $\mathrm{P}(\xi_n \in J_{\alpha,n}) \to 1 - \alpha$ as $n \to \infty$. As a conclusion, $P(x_{\mathrm{param},\,n} \notin CI_{\alpha,n} \mid \mathcal{H}_0) \to \alpha$ as $n \to \infty$.

Proof of Theorem 3

From Theorem 1, and under $\mathcal{H}_1$,

$$x_{\mathrm{param},\,n} - \widehat{x}_{\mathrm{ET},\,n} = x_{\mathrm{param},\,n} - x_{\mathrm{ET},\,1,n} + \sigma_1(u_{1,n}) \ln(r_n) m_n^{-1/2}\ \xi_n$$

where $\xi_n \xrightarrow{d} \xi \sim N(0,1)$. Therefore, in view of (11),

$$P(x_{\mathrm{param},\,n} \notin CI_{\alpha,n} \mid \mathcal{H}_1) = P\left(\xi_n \notin J'_{\alpha,n}\right), \tag{16}$$

where $J'_{\alpha,n} = [\omega_n\psi_n + \phi_n \pm z_\alpha\widehat{\sigma}_n/\sigma_1(u_{1,n})]$. Now, since $m_n \to \infty$ and $m_n/n \to 0$, under $\mathcal{H}_1$, we have $\widehat{\sigma}_n/\sigma_1(u_{1,n}) \xrightarrow{P} 1$ as $n \to \infty$ [4]. For $\eta > 0$, introducing $K'_{\alpha,\,n} = [\omega_n\psi_n + \phi_n \pm z_\alpha(1+\eta)]$, leads to $P(J'_{\alpha,n} \subset K'_{\alpha,\,n}) \to 1$ as $n \to \infty$. From Lemma 4.6 and Lemma 4.7, $K'_{\alpha,\,n}$ can be rewritten as $K'_{\alpha,\,n} = [\phi_n(1+\varepsilon_n) \pm z_\alpha(1+\eta)]$, with $\varepsilon_n = \omega_n\psi_n/\phi_n \to 0$ as $n \to \infty$. It follows that $\forall \eta > 0$, $\mathrm{P}(\xi_n \notin K'_{\alpha,\,n}) \to 1$ as $n \to \infty$, since $|\phi_n| \to \infty$. Remarking that

$$\begin{aligned} P(\xi_n \notin K'_{\alpha,\,n}) &= P(\{\xi_n \notin J'_{\alpha,n}\} \cap \{J'_{\alpha,n} \subset K'_{\alpha,\,n}\}) + P(\{\xi_n \notin J'_{\alpha,n}\} \cap \{J'_{\alpha,n} \not\subset K'_{\alpha,\,n}\}) \\ &\leq P(\xi_n \notin J'_{\alpha,n}) + P(J'_{\alpha,n} \not\subset K'_{\alpha,\,n}), \end{aligned}$$

it results that $1 \geq P(\xi_n \notin J'_{\alpha,n}) \geq P(\xi_n \notin K'_{\alpha,\,n}) - P(J'_{\alpha,n} \not\subset K'_{\alpha,\,n}) \to 1$ as $n \to \infty$. Therefore $P(\xi_n \notin J'_{\alpha,n}) \to 1$ as $n \to \infty$ and (16) concludes the proof.

Appendix: Proof of lemmas.

Proof of Lemma 4.1

This is a classical result. Remark that $S_n = m_n^{-1/2}(N_n - m_n)$ is the sum of a triangular array of independent and centered random variables $(Y_{i,\,n})$:

$$Y_{i,\,n} = m_n^{-1/2}(\mathbb{I}\{X_i > u_n\} - m_n/n).$$

Since $\mathrm{Var}(S_n) \to 1$ as $n \to \infty$, Lyapunov's condition reduces to

$$\sum_{i=1}^{n} \mathrm{E}(|Y_{i,\,n}|^3) \to 0$$

as $n \to \infty$. Straightforward calculations show that the above sum is $O(m_n^{-1/2})$.

Proof of Lemma 4.2

Step 1. We first prove that $N_n^{1/2}(\widehat{\sigma}_n/\sigma(u_n) - \mu(u_n)) \xrightarrow{d} N(0,1)$ conditionally on $N_n = k_n$, where (k_n) is a sequence tending to infinity. To this end, we introduce the empirical process of the normalized excesses $\mathbb{A}_n^\sharp(y) = N_n^{-1}\sum_{j=1}^{N_n} \mathbb{I}\{Y_j \le y\sigma(u_n)\}$. We have

$$\widehat{\sigma}_n/\sigma(u_n) = \int_0^\infty y\, d\mathbb{A}_n^\sharp(y) = \int_0^\infty (1 - \mathbb{A}_n^\sharp(y))\, dy. \tag{17}$$

Collecting (9) and (17), we obtain

$$N_n^{1/2}(\widehat{\sigma}_n/\sigma(u_n) - \mu(u_n)) = -\int_0^\infty N_n^{1/2}(\mathbb{A}_n^\sharp(y) - F_{u_n}^\sharp(y))\, dy.$$

Conditionally on $N_n = k_n$, the stochastic process $N_n^{1/2}(\mathbb{A}_n^\sharp - F_{u_n}^\sharp)$ can be written as $\mathbb{B}_{k_n} \circ F_{u_n}^\sharp$, where $\mathbb{B}_{k_n}$ is the empirical process built from k_n independent uniform random variables on $[0, 1]$. From Pickands' theorem, for all sequence (k_n) tending to infinity, $F_{u_n}^\sharp$ converges uniformly to G, the cumulative distribution function of the Exp(1) distribution. Consequently, $\mathbb{B}_{k_n} \circ F_{u_n}^\sharp$ converges in distribution to $\mathbb{B} \circ G$ as $n \to \infty$, where $\mathbb{B}$ is a Brownian bridge on $[0, 1]$. Therefore, conditionally on $N_n = k_n$,

$$N_n^{1/2}(\widehat{\sigma}_n/\sigma(u_n) - \mu(u_n)) \xrightarrow{d} -\int_0^\infty \mathbb{B}(G(y))\, dy$$

as $n \to \infty$, for all sequence (k_n) tending to infinity. For symmetry reasons, we have

$$-\int_0^\infty \mathbb{B}(G(y))\, dy \stackrel{d}{=} -\int_0^\infty \mathbb{B}(1 - G(y))\, dy \stackrel{d}{=} \int_0^1 t^{-1}\mathbb{B}(t)\, dt \stackrel{d}{=} \int_0^1 \ln t\, d\mathbb{B}(t),$$

where $t = 1 - G(y)$. As a preliminary conclusion, we have shown that conditionally on $N_n = k_n$, $N_n^{1/2}(\widehat{\sigma}_n/\sigma(u_n) - \mu(u_n)) \xrightarrow{d} N(0,1)$ as $n \to \infty$ for all sequence (k_n) tending to infinity.

Step 2. Let us denote $Z_n = N_n^{1/2}(\widehat{\sigma}_n/\sigma(u_n) - \mu(u_n))$. In the first step of the proof, we have shown that for each Borel subset B, $P(Z_n \in B | N_n = k_n) \to P(Z \in B)$ as $n \to \infty$, where $Z \sim N(0, 1)$. The proof of $P(Z_n \in B) \to P(Z \in B)$ as $n \to \infty$ is based on Lebesgue's convergence theorem.

Proof of Lemma 4.3

The survival distribution function of the normalized excess $\bar{F}_{u_n}^\sharp$ can be rewritten in terms of the function V to obtain

$$\mu(u_n) = \int_0^\infty \exp\left(V^{-1}(u_n) - V^{-1}(u_n + y\sigma(u_n))\right)\, dy.$$

Setting $x = V^{-1}(u_n + y\sigma(u_n)) - V^{-1}(u_n)$ and remarking that $\int_0^\infty e^{-x}\,dx = 1$ yields

$$\mu(u_n) - 1 = \int_0^\infty \left(\frac{V'(x + V^{-1}(u_n))}{V'(V^{-1}(u_n))} - 1 \right) e^{-x}\,dx.$$

For the sake of simplicity, define $z_n = V^{-1}(u_n)$. Then,

$$\frac{\mu(u_n) - 1}{a(u_n)} = \int_0^\infty \frac{1}{A(e^{z_n})} \left(\frac{V'(x + z_n)}{V'(z_n)} - 1 \right) e^{-x}\,dx.$$

The second order condition **(A2)** implies that

$$\frac{1}{A(e^{z_n})} \left(\frac{V'(x + z_n)}{V'(z_n)} - 1 \right) \to \int_0^x e^{\rho s}\,ds \tag{18}$$

as $n \to \infty$ (see [5] and [16, 14]). Moreover, for n sufficiently large and all $0 < \varepsilon < 1$ arbitrarily small, we have

$$\frac{1}{|A(e^{z_n})|} \left| \frac{V'(x + z_n)}{V'(z_n)} - 1 \right| e^{-x} \le C e^{(\varepsilon - 1)x} \tag{19}$$

for some positive constant C. This result is established in [15], Proposition 1. This kind of Potter bounds has also been given in [5], Corollary 2.4. From (18) and (19), we conclude that

$$\frac{\mu(u_n) - 1}{a(u_n)} \to \int_0^\infty \int_0^x e^{\rho s - x}\,ds\,dx = \frac{1}{1 - \rho}$$

as $n \to \infty$. Consequently, **(A4)** implies that $m_n^{1/2}(\mu(u_n) - 1) \to 0$ as $n \to \infty$.

Proof of Lemma 4.5

Let us introduce $-\ln \zeta' = \kappa_2 - \ln \zeta$ and $-\ln \zeta'' = \kappa_2 - \ln \zeta - \kappa_3(\kappa_2 + \kappa_1 \ln x)$. We can note that, since $\zeta \to 0$, since κ_1, κ_2, and κ_3 are bounded and since $\ln x$ is neglictible with respect to $\ln \zeta$, we have $\ln \zeta' \sim \ln \zeta$ and $\ln \zeta'' \sim \ln \zeta$. We consider two cases:

- If $V \in \mathcal{C}^1_\theta \cup \mathcal{C}^2$, then V' and V'' are smooth regularly varying functions and the result is well-known. Let us remark that if $V \in \mathcal{C}^1_{1,\infty}$ then $V''/V' = 0$ and equivalence (10) is an equality.
- Suppose $V \in \mathcal{C}^3_\theta$. We thus have $V = \exp g$ where $g \in \mathcal{SR}_\theta$. Our goal is to show that

$$\frac{V''(-\ln \zeta'')}{V''(-\ln \zeta)} \frac{V'(-\ln \zeta)}{V'(-\ln \zeta')} \sim 1.$$

  The proofs of $V''(-\ln \zeta'')/V''(-\ln \zeta) \sim 1$ and $V'(-\ln \zeta)/V'(-\ln \zeta') \sim 1$ are similar. Consider for instance the first term:

$$\begin{aligned} \frac{V''(-\ln \zeta'')}{V''(-\ln \zeta)} &= \frac{(g'' + g'^2)(-\ln \zeta'')}{(g'' + g'^2)(-\ln \zeta)} \exp(g(-\ln \zeta'') - g(-\ln \zeta)) \\ &\sim \exp(g(-\ln \zeta'') - g(-\ln \zeta)), \end{aligned}$$

since $g'' + {g'}^2 \in SR_{2\theta-2}$ and $-\ln\zeta'' \sim -\ln\zeta$. A Taylor expansion of g shows that there exists $\kappa \in]0,1[$ such that

$$\frac{V''(-\ln\zeta'')}{V''(-\ln\zeta)} \sim \exp(\ln(\zeta/\zeta'')g'(-\ln\zeta'' - \kappa\ln(\zeta/\zeta''))).$$

Remarking that $\ln(\zeta/\zeta'') = \kappa_2 - \kappa_3(\kappa_2 + \kappa_1 \ln x)$ is independent from ζ and that $-\ln\zeta'' \to \infty$ as $\zeta \to 0$ yields $g'(-\ln\zeta'' - \kappa\ln(\zeta/\zeta'')) \to 0$ as $\zeta \to 0$, since g' is a smooth regularly varying function with negative index.

Proof of Lemma 4.6

Let us introduce $a_n = \ln(n/m_n)$ and $b_n = \ln(1/p_n)$. In view of Remark 2.3, we have $\omega_n = o(\Lambda_n)$ where we have defined $\Lambda_n = \sqrt{m_n}(a_n - b_n)V_0''(\varrho_n)/V_0'(a_n)$ with $\varrho_n \in [b_n, a_n]$. Let us consider first the case $V_0 \in \mathcal{C}^1_{1,\infty}$. We thus have $\Lambda_n = 0$. In all the other cases, we suppose $p = p' = 1$ and we write $\varrho_n = a_n + \kappa(b_n - a_n)$ with $\kappa \in [0,1]$. Thus, setting $\kappa_2 = 0$, $\kappa_1 = \kappa_3 = 1$, $\ln x = \kappa(a_n - b_n)$, and $\ln\zeta = -a_n$ yields $\ln x/\ln\zeta = \kappa(b_n/a_n - 1) \to 0$ as $n \to \infty$. Lemma 4.5 entails that

$$\Lambda_n \sim \sqrt{m_n}(a_n - b_n)V_0''(a_n)/V_0'(a_n),$$

and Lemma 4.4 concludes the proof.

Proof of Lemma 4.7

It is easily seen that $\psi_n = \Omega(a_n)\Omega_0(a_n)/\Omega_1(a_n)$ and

$$\phi_n = \sqrt{m_n}\,\frac{\Omega(a_n)}{\Omega_1(a_n)}\left[\frac{a_n}{b_n - a_n}\left(\frac{1}{\Omega(a_n)} - 1\right) + \frac{\Omega_1(a_n)}{\Omega(a_n)} - \Omega_0(a_n)\right],$$

which leads to

$$\frac{\phi_n}{\psi_n} = \frac{\sqrt{m_n}}{\Omega_0(a_n)}\left[\frac{a_n}{b_n - a_n}\left(\frac{1}{\Omega(a_n)} - 1\right) + \frac{\Omega_1(a_n)}{\Omega(a_n)} - \Omega_0(a_n)\right].$$

Remark that (8) ensures that $\liminf a_n/(b_n - a_n) > 0$. We only prove (i), the proofs of (ii) and (iii) being similar. Introducing $H_0 \in \mathcal{C}^1_\theta$, we have $\Omega_0(a_n) \to 1/\theta$ as $n \to \infty$. Three cases appear:

- $V_1 \in \mathcal{C}^1_{\theta'}$, $\theta' \neq \theta$. Suppose for instance $\theta < \theta'$. It follows that $\Omega_1(a_n) \to 1/\theta'$ as $n \to \infty$ and $\Omega \in \mathcal{SR}_{1/\theta - 1/\theta'}$ with $1/\theta - 1/\theta' > 0$ leading to $\Omega(a_n) \to \infty$. This yields

$$\phi_n \sim -\theta'\sqrt{m_n}\Omega(a_n)\left[\frac{a_n}{b_n - a_n} + \frac{1}{\theta}\right] \to -\infty$$

  and

$$\frac{\phi_n}{\psi_n} \sim -\theta\sqrt{m_n}\left[\frac{a_n}{b_n - a_n} + \frac{1}{\theta}\right] \to -\infty$$

  as $n \to \infty$. The case $\theta > \theta'$ is similar.

- $V_1 \in \mathcal{C}^2$. In this case, $\Omega_1(a_n) \to 0$ as $n \to \infty$ and $\Omega \in \mathcal{SR}_{1/\theta}$ which is the previous situation.
- $V_1 \in \mathcal{C}^3_\theta$. We have $\Omega_1(a_n) \to +\infty$ as $n \to \infty$ and $\Omega(a_n) = V_0(a_n)\exp(-g(a_n))$ with $g \in \mathcal{SR}_\theta$ leading to $\Omega(a_n) \to 0$ as $n \to \infty$. This yields

$$\phi_n \sim \sqrt{m_n}\left[\frac{a_n}{b_n - a_n}\frac{1}{\Omega_1(a_n)} + 1\right] \to \infty$$

and

$$\frac{\phi_n}{\psi_n} \sim \theta\frac{\sqrt{m_n}}{\Omega(a_n)}\left[\frac{a_n}{b_n - a_n} + \Omega_1(a_n)\right] \to \infty$$

as $n \to \infty$.

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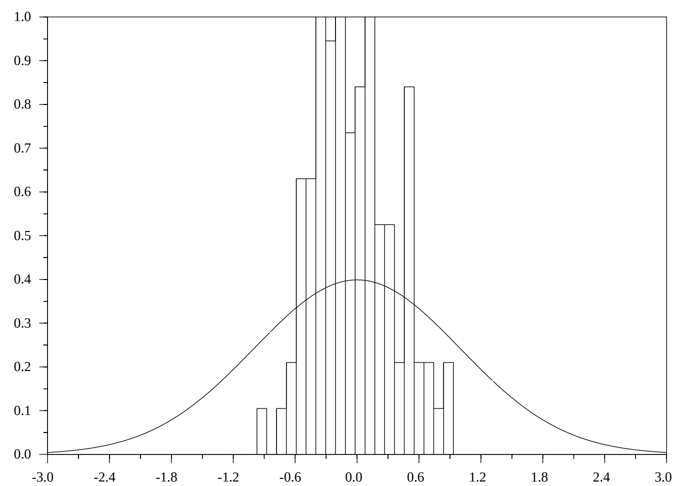
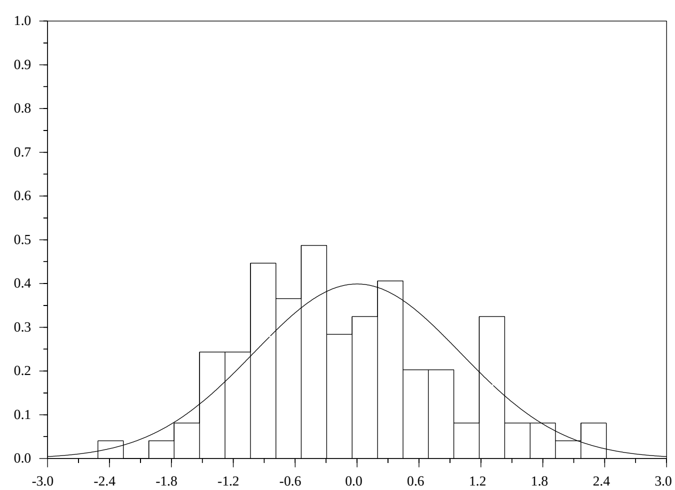


(a) Normalization (6)



(b) Normalization (5)

Figure 1: Comparison of the $N(0, 1)$ density with the empirical distribution obtained by both normalizations.

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